IMM 364
January 1968

Courant Institute of
Mathematical Sciences

Interacting Motion of Rectilinear Geostrophic Vortices
I. Linear Stability Analysis

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Prepared under Contract Nonr-285(55)
with the Office of Naval Research
NR 062-160 and Contract AT(30-1)-1480
with the U.S. Atomic Energy Commission

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New York University

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INTERACTING MOTION OF RECTILINEAR GEOSTROPHIC VORTICES

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This report represents results obtained at the Courant Institute of Mathematical Sciences, New York University. This work was supported in part by the Office of Naval Research, Contract Nonr-285(55), and in part by the U.S. Atomic Energy Commission, Contract AT(30-1)-1480.

ABSTRACT

The kinematic motion of $(N + 1)$ geostrophic (Bessel) vortices is studied. Initially the vortices are positioned near a uniformly rotating equilibrium configuration which consists of N vortices of equal strength γ, equally spaced on a circle of radius a, and one vortex of strength γ_o at the center of the circle. One length scale κ^{-1} is associated with each of the vortices, the limiting case $\kappa = 0$ being the classical logarithmic vortex. The exponential stability of this equilibrium configuration is studied over the range of the three parameters: $N \geq 2$, $0 \leq (\kappa a) < \infty$ and $- \infty < (\gamma_o/\gamma) < \infty$. The need for a supplementary nonlinear study is discussed.

1. Continuum and Discrete Description of Geostrophic Motion.

The first systematic study of geostrophic vortex motion was begun by H. J. Stewart [1,2]*, using primitive computers by present-day standards. By evoking some simple, but rather convincing, physical arguments he represented the large-scale closed isobaric systems of the atmosphere by discrete rectilinear geostrophic (Bessel) vortices. Our paper is an extension of this earlier work.

In more recent years with the advent of large electronic computers, formally more systematic derivations of the approximate non-linear equations describing geostrophic motion have been found, e.g. by Obukhov [3] and Morikawa [4]. Meteorologists have applied such equations to short range weather forecasting studies using continuum (or smooth) initial data.

During the past few years we have been studying, mainly by numerical methods, a number of different types of initial value problems, i.e. 1) purely continuum data, 2) a combination of smooth and discrete (vortices) data, and 3) purely discrete data. A primary interest is to understand the

*The existence of Stewart's early work which apparently has been ignored in the meteorological literature, was made known to one of us by S. C. Lowell.

non-linear interacting development of the motion. We have done more work on 1) and 3) than on 2) which requires the most computer space*. In this paper we describe in part some of our work on 3): specifically, we study the interacting motion of N rectilinear geostrophic vortices of equal strength γ initially distributed approximately with equal spacing near (or on) a circle of radius a and of a single vortex of strength γ_0 placed initially near (or at) the center of the circle. All N + 1 vortices have the same horizontal length scale κ^{-1} where the limiting case $\kappa = 0$ corresponds to potential (logarithmic) vortices. This paper consists of two principal parts: I) linearized stability analysis, an eigenvalue problem, and II) numerical computation of the non-linear motions in sufficient detail to interpret the stability results. The study of linearized stability for the special case of $\kappa = 0$, $\gamma_0 = 0$ goes back many years to an early essay of J. J. Thomson [8] who was interested in developing a vortex model of an atom (unsuccessful).

A single rectilinear geostrophic vortex in an

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*Some of that part of 2) related to describing hurricane trajectories, using a single vortex to represent the hurricane, has been reported in [5] at a Symposium in Tokyo in 1960; this numerical work was carried out with E. Isaacson and D. A. Levine [6]. Some results, comparing the difference in numerical solutions between the use of smooth data and a combination of smooth data interacting with a single vortex, were presented at a Symposium in Mexico City in 1963 [7].

unconfined space is given by the stream function

$$\psi_o = \frac{\gamma}{2\pi} K_o(\kappa|r - r_o|) \tag{1.1}$$

where the stream function ψ for any geostrophic motion (in our sense) is related to the velocity components by

$$(u,v) = (-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x}) \tag{1.2}$$

which states that this approximate two-dimensional description of large-scale nearly horizontal atmospheric motion is divergence-free; but in addition (1.2) states that the flow is geostrophic (and hydrostatic, as well) to this order of approximation, i.e. the inertia term in the horizontal momentum equations is small compared to the pressure gradient and coriolis terms. The distance $|r - r_o| = [(x - x_o)^2 + (y - y_o)^2]^{1/2}$ where the vortex position (x_o, y_o) may be a function of time if there are other elements in the field of motion. K_o is the Bessel function of the second kind (zeroth order), singular at the origin.

For the problem which we consider with (N + 1) vortices, there are <u>three dimensionless parameters</u>: N (discrete), (κa) (continuous) and (γ_o/γ) (continuous) where $N \geq 2$, $0 \leq (\kappa a) < \infty$ and $-\infty < (\gamma_o/\gamma) < \infty$. In this paper we only consider the motion in an <u>unconfined space</u> although flow within (or outside of) a finite closed boundary or

bounded by a semi-infinite wall would be of interest*; the study of such flows are harder. We also restrict ourselves in §3 to the usual linearized stability problem, i.e. a normal-mode type of analysis for small perturbations of the position (x_k, y_k) of the vortices. Because of the number of relevant parameters this much, combined with the non-linear numerical calculations, yields a considerable amount of information. Other interesting stability problems are 1) perturbations of κ_k and 2) perturbations of γ_k; these problems may be considered in the future.

We make a few remarks about the possible applicability of our results to the description of actual fluid motions. Since the rectilinear geostrophic vortices are derived from a tangent plane approximation of atmospheric motion over the earth, some discretion must be used in trying to apply them directly to the study of global motions in which there are large amplitude meridional (north-south) motions. Although the original motivation for these studies has come from meteorological considerations, we believe our results may be relevant in the description of other fluid

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*For example, Onsager [9] has studied the problem of clustering of N logarithmic vortices of arbitrary strength γ_i moving inside of a closed boundary.

mechanical phenomena* in which the forces of gravitation and rotation are in approximate balance. The study of kinematic instability may be relevant to the turbulence problem.

In a later report we shall present the results of our nonlinear numerical computations, which will clarify the range of validity of the linearized stability analysis.

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*Recently geostrophic vortices have been used in the study of superconductivity [10].

2. Equations of Motion and Equilibrium.

For this (N + 1) - vortex problem, the positions of the N circle vortices are best described in polar coordinates (r_i, θ_i) and the position of the center vortex, in cartesian coordinates (x_o, y_o). The non-linear (kinematic) motion of the k^{th} circle vortex in polar coordinates (r_k, θ_k) satisfies

$$(2.1) \qquad \dot{r}_k = -\kappa^2 \sum_{\substack{i \neq k \\ i=1}}^{N} \left[r_i \sin \theta_{ki} \frac{K_1(\rho_{ki})}{\rho_{ki}} \right]$$

$$- \gamma_o \kappa^2 (x_o \sin \theta_k - y_o \cos \theta_k) \frac{K_1(\rho_{ko})}{\rho_{ko}}$$

$$(2.2) \qquad r_k \dot{\theta}_k = \kappa^2 \sum_{\substack{i \neq k \\ i=1}}^{N} \left[(r_k - r_i \cos \theta_{ki}) \frac{K_1(\rho_{ki})}{\rho_{ki}} \right]$$

$$+ \gamma_o \kappa^2 \left[r_k - (x_o \cos \theta_k + y_o \sin \theta_k) \right] \frac{K_1(\rho_{ko})}{\rho_{ko}}$$

where $(\dot{\ })$ means time derivative; K_1 is the Bessel function of the second kind (first order); and

$$\rho_{ki} = \kappa (r_k^2 + r_i^2 - 2 r_k r_i \cos \theta_{ki})^{1/2} , \quad \theta_{ki} = (\theta_k - \theta_i)$$

$$\rho_{ko} = \kappa \left[r_k^2 - 2 r_k (x_o \cos \theta_k + y_o \sin \theta_k) + (x_o^2 + y_o^2) \right]^{1/2} .$$

The non-linear motion of the center vortex in cartesian coordinates (x_o, y_o) is described by

$$(2.3) \qquad \dot{x}_o = -\kappa^2 \sum_{i=1}^{N} \left[(y_o - r_i \sin \theta_i) \frac{K_1(\rho_{oi})}{\rho_{oi}} \right]$$

$$(2.4) \qquad \dot{y}_o = \kappa^2 \sum_{i=1}^{N} \left[(x_o - r_i \cos \theta_i) \frac{K_1(\rho_{oi})}{\rho_{oi}} \right]$$

where

$$\rho_{oi} = \kappa \left[r_i^2 - 2r_i(x_o \cos \theta_i + y_o \sin \theta_i) + (x_o^2 + y_o^2) \right]^{1/2}.$$

The following dimensional normalizations have been made: lengths are normalized by the circle radius a; time is normalized by $(2\pi a^2/\gamma)$; and the center vortex strength γ_o is normalized by γ. The 2(N + 1) Eqs. (2.1) - (2.4) state that the velocity of a particular vortex, say the k^{th} one, is the sum of the field velocities of the remaining vortices evaluated at the k^{th} vortex position, each field velocity being directed normal to the distance vector from each vortex to the k^{th} vortex; changing the sign of the strength γ_o (or γ) reverses the direction of rotation of the vortex. In this dynamically balanced system, each vortex moves slowly, like a massless particle, in the velocity field of the other vortices.

2.1 Equilibrium Motion.

For N vortices of the same strength γ equally spaced on a circle and one vortex of strength γ_o at the center of the circle, the circle vortices move with uniform angular velocity Ω around the center vortex. For this equilibrium

configuration the circle vortices satisfy the following conditions:

$$r_k = 1 \ ; \qquad \dot{r}_k = 0$$

$$\theta_k = w_k \ , \qquad \dot{\theta}_k = \dot{w}_k = \Omega \ , \quad \text{const.}$$

where

$$w_k = \Omega t + w_k^o \ ,$$

and the superscripts denote the initial equilibrium values

$$w_k^o = \frac{2\pi k}{N} \ .$$

The center vortex satisfies

$$x_o = y_o = 0 \ ; \quad \dot{x}_o = \dot{y}_o = 0 \ .$$

Inserting these conditions in Eqs. (2.1) - (2.4) we obtain for the circle vortices

$$\sum_{\substack{i \neq k \\ i=1}}^{N} \left[\sin w_{ki} \frac{K_1(\sigma_{ki})}{\sigma_{ki}} \right] = 0 \tag{2.5}$$

and

$$\Omega = \Omega_o + \gamma_o \omega_o \tag{2.6}$$

$$\Omega_o = \frac{1}{2} \sum_{\substack{i \neq k \\ i=1}}^{N} \left[\sigma_{ki} K_1(\sigma_{ki}) \right]$$

$$\omega_o = \kappa K_1(\kappa)$$

where $w_{ki} = (w_k - w_i) = (w_k^o - w_i^o)$ and $\sigma_{ki} = \kappa\left[2(1 - \cos w_{ki})\right]^{1/2}$. The center vortex equations yield the conditions

$$(2.7) \qquad \sum_{i=1}^{N} \sin w_i = 0 \ , \quad \sum_{i=1}^{N} \cos w_i = 0 \ .$$

For example, the equilibrium configuration for N = 8 is shown in Fig. 1.

3. Linearized Stability Analysis of N Vortices on a Circle and One Vortex at the Center.

The linearized equations of motion are obtained in the usual way. We make a perturbation expansion of the position vector $(r_i, \theta_i;\ x_o, y_o)$ on the equilibrium solution; the perturbation parameter is $\varepsilon \ll 1$. For the k^{th} circle vortex

$$(3.1) \qquad r_k = 1 + \varepsilon r_k^{(1)} + O(\varepsilon^2)\ , \quad \dot{r}_k = \varepsilon \dot{r}_k^{(1)} + O(\varepsilon^2)$$

$$(3.2) \qquad \theta_k = w_k + \varepsilon \theta_k^{(1)} + O(\varepsilon^2)\ , \quad \dot{\theta}_k = \Omega + \varepsilon \dot{\theta}_k^{(1)} + O(\varepsilon^2)$$

where the equilibrium angular velocity $\Omega = \dot{w}_k$ is given by (2.6). For the center vortex

$$(3.3) \qquad x_o = \varepsilon x_o^{(1)} + O(\varepsilon^2)\ , \quad \dot{x}_o = \varepsilon \dot{x}_o^{(1)} + O(\varepsilon^2)$$

$$(3.4) \qquad y_o = \varepsilon y_o^{(1)} + O(\varepsilon^2)\ , \quad \dot{y}_o = \varepsilon \dot{y}_o^{(1)} + O(\varepsilon^2)\ .$$

The expanded solution (3.1) - (3.4) is put into (2.1) - (2.4), retaining only first order terms in ε. The presence of the center vortex introduces periodic coefficients into the resulting system of linear differential equations for $(r_k^{(1)}, \theta_k^{(1)};\ x_o^{(1)}, y_o^{(1)})$. However, this apparent difficulty can be eliminated by transforming the equations to a frame of reference rotating with the equilibrium angular velocity Ω,

$$\tilde{x}_o^{(1)} = x_o^{(1)} \cos \Omega t + y_o^{(1)} \sin \Omega t$$

$$\tilde{y}_o^{(1)} = - x_o^{(1)} \sin \Omega t + y_o^{(1)} \cos \Omega t\ .$$

The resulting system of $2(N + 1)$ linear differential equations for $(r_k^{(1)}, \theta_k^{(1)};\ \tilde{x}_\odot^{(1)}, \tilde{y}_\odot^{(1)})$ with <u>constant coefficients</u> are

$$(3.5) \qquad \dot{r}_k^{(1)} + \sum_{\substack{i \neq k \\ i=1}}^{N} A_{ki} r_i^{(1)} - \sum_{\substack{i \neq k \\ i=1}}^{N} B_{ki} \theta_{ki}^{(1)} + \gamma_o \omega_o (\tilde{x}_\odot^{(1)} \sin w_k^o - \tilde{y}_\odot^{(1)} \cos w_k^o) = 0$$

$$(3.6) \qquad \dot{\theta}_k^{(1)} + (\Omega + C_o + \gamma_o D_o) r_k^{(1)} + \sum_{\substack{i \neq k \\ i=1}}^{N} C_{ki} r_i^{(1)} + \sum_{\substack{i \neq k \\ i=1}}^{N} A_{ki} \theta_i^{(1)} - \gamma_o D_o (\tilde{x}_\odot^{(1)} \cos w_k^o + \tilde{y}_\odot^{(1)} \sin w_k^o) = 0$$

$$(3.7) \qquad \dot{\tilde{x}}_\odot^{(1)} + D_o \sum_{i=1}^{N} r_i^{(1)} \sin w_i^o - \omega_o \sum_{i=1}^{N} \theta_i^{(1)} \cos w_i^o - \left[(\omega_o + D_o) \sum_{i=1}^{N} \sin w_i^o \cos w_i^o\right] \tilde{x}_o^{(1)} - \left[(\omega_o + D_o) \sum_{i=1}^{N} \sin^2 w_i^o - N\omega_o + \Omega\right] \tilde{y}_\odot^{(1)} = 0$$

$$(3.8) \qquad \dot{\tilde{y}}_o^{(1)} - D_o \sum_{i=1}^{N} r_i^{(1)} \cos w_i^o - \omega_o \sum_{i=1}^{N} \theta_i^{(1)} \sin w_i^o \quad \left[(\omega_o + D_o) \sum_{i=1}^{N} \cos^2 w_i^o - N\omega_o + \Omega\right] \tilde{x}_\odot^{(1)} + \left[(\omega_o + D_o) \sum_{i=1}^{N} \sin w_i^o \cos w_i^o\right] \tilde{y}_\odot^{(1)} = 0$$

where

$$A_{ki} = \frac{-\kappa^2}{2} \left[\sin w_{ki}^o \; K_o(\sigma_{ki}) \right]$$

$$B_{ki} = \frac{\kappa^2}{2} \left[(1 + \cos w_{ki}^o) \; K_o(\sigma_{ki}) + \frac{2K_1(\sigma_{ki})}{\sigma_{ki}} \right]$$

$$C_{ki} = \frac{\kappa^2}{2} \left[(1 - \cos w_{ki}^o) \; K_o(\sigma_{ki}) + \frac{2K_1(\sigma_{ki})}{\sigma_{ki}} \right]$$

$$C_o = \frac{\kappa^2}{2} \sum_{\substack{i \neq k \\ i=1}}^{N} \left[(1 - \cos w_{ki}^o) \; K_o(\sigma_{ki}) - 2 \cos w_{ki}^o \; \frac{K_1(\sigma_{ki})}{\sigma_{ki}} \right]$$

$$\sigma_{ki} = \kappa \left[2(1 - \cos w_{ki}^o) \right]^{1/2} , \quad w_{ki}^o = \frac{2\pi}{N} (k - i)$$

$$\omega_o = \kappa K_1(\kappa) , \quad D_o = \kappa^2 \left[K_o(\kappa) + \frac{K_1(\kappa)}{\kappa} \right]$$

Eqs. (3.5) - (3.8) are valid for $N \geq 2$. For $N \geq 3$ the center vortex equations (3.7) and (3.8) simplify because of increased geometrical symmetry expressed by the additional conditions*

$$(3.9) \quad \sum_{i=1}^{N} \sin 2 w_i = 2 \sum_{i=1}^{N} \sin w_i \cos w_i = 0 ,$$

$$\sum_{i=1}^{N} \cos 2w_i = \sum_{i=1}^{N} (\cos^2 w_i - \sin^2 w_i) = 0$$

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*A. Troesch has informed us that the conditions (2.7) and (3.9) are included in a function theoretic result due to Veska.

and the identity

$$(3.10) \qquad \sum_{i=1}^{N} (\sin^2 w_i + \cos^2 w_i) = N$$

3.1 Computation of Eigenvalues.

Since the linearized equations of motion (3.5) - (3.8) have constant coefficients the usual exponential stability analysis is possible. The displacement $(r_k^{(1)}, \theta_k^{(1)};\ \tilde{x}_o, \tilde{y}_o)$ from the equilibrium position can be written as a sum of eigensolutions in the form exp. $(\lambda_\ell t)$ where $\ell = 1,2,\ldots 2(N+1)$. The motion is unstable with respect to the equilibrium position if any eigenvalue λ_ℓ has a positive real part.

If the center vortex is absent $(\gamma_o = 0)$, there is a high degree of symmetry and the eigenvalue calculation can be reduced to solving a quadratic equation. For $\gamma_o = 0$, the reduced Eqs. (3.5) and (3.6) take the matrix form

$$(3.11) \qquad \begin{pmatrix} A & B \\ C & A \end{pmatrix} \begin{pmatrix} r_j^{(1)} \\ \theta_j^{(1)} \end{pmatrix} = \lambda_\ell \begin{pmatrix} r_j^{(1)} \\ \theta_j^{(1)} \end{pmatrix}$$

where $j = 1,2,\ldots N$, $\ell = 1,2,\ldots 2N$ and the sub-matrices A, B and C are each N x N right-circulant matrices. For example, the elements of A for N = 4 are

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{14} & a_{11} & a_{12} & a_{13} \\ a_{13} & a_{14} & a_{11} & a_{12} \\ a_{12} & a_{13} & a_{14} & a_{11} \end{pmatrix} .$$

The eigenvectors of a circulant matrix are independent of the matrix elements and are given by

$$(1, \omega^{j}, \omega^{2j}, \ldots, \omega^{(n-1)j})$$

where

$$\omega = e^{i(\frac{2\pi}{N})}, \quad i = (-1)^{1/2} .$$

Then the sub-matrices A, B and C all have the same eigenvectors and hence commute; their respective eigenvalues are

(3.12) $$a_j = a_{11} + a_{12}\omega^{j} + \ldots + a_{1N}\omega^{(N-1)j}, \quad j = 1,2,\ldots,N$$

(3.13) $$b_j = b_{11} + b_{12}\omega^{j} + \ldots + b_{1N}\omega^{(N-1)j}$$

(3.14) $$c_j = c_{11} + c_{12}\omega^{j} + \ldots + c_{1N}\omega^{(N-1)j} .$$

The eigenvalues λ_ℓ in (3.11) are given by

(3.15) $$\lambda_\ell = a_j \pm (b_j c_j)^{1/2}$$

which agrees with Stewart [1].

In the general case when the center vortex is present

($\gamma_0 \neq 0$), the circulant property is destroyed and the eigenvalue calculation for the resulting $2(N + 1) \times 2(N + 1)$ matrix is carried out by an iteration procedure devised by B. Parlett [11]. The ($\gamma_0 = 0$) case has been used to test the accuracy of this iteration method.

3.2 N Vortices on a Circle ($\gamma_0 = 0$).

This case was considered by Thomson [9] for logarthmic vortices ($\kappa = 0$) and Stewart [1,2] for the geostrophic K_0-vortices ($\kappa > 0$).

A physically significant transition region, depending on both κ and N, is found separating stable and unstable configurations. In Fig. 2 the variation of the maximum real eigenvalues with respect to κ for $2 < N \leq 9$ is shown. For $N = 6$ the circle configuration is stable for $0 \leq \kappa \leq 1.288$ and unstable* for $\kappa \geq 1.289$. For $N \geq 7$ the circle configuration is unstable for all κ; except $\kappa = 0$ is a neutral** stability point for $N = 7$. For $2 \leq N \leq 5$ the circle configuration is stable for all κ.

In Fig. 3 the variation of the imaginary eigenvalues (normal or eigenfrequencies) with respect to κ for $N = 6$ is

*This corrects Stewart's result that $N = 6$ is stable for all κ.

**This corrects Thomson's slightly positive eigenvalue.

shown. The equilibrium frequency* Ω_0 is marked. All of the normal frequencies decrease with increasing κ. However, it is physically more meaningful to compare all frequencies with Ω_0 (or Ω). Since in general Ω decreases more rapidly than the other eigenfrequencies with increasing κ, vortices with larger κ behave like particles with harder potentials in dynamical systems. In addition there are double zero eigenvalues for all values of N and κ; these are discussed in §3.6.

3.3 (N + 1) Vortices ($\gamma_0 = 1$).

This case, with N vortices on a circle and a center vortex of the same strength as each of the circle vortices, illustrates some of the main features of the general case ($\gamma_0 \geq 0$). Comparison with the case ($\gamma_0 = 0$) is particularly instructive.

Similar to the case without a center vortex, a physically significant transition region is found separating stable and unstable configurations for $\gamma_0 = 1$. In Fig. 4 the variation of the maximum real eigenvalues with respect to κ for $2 \leq N \leq 10$ is shown. For N = 8 the (N + 1)-vortex configuration is stable for $0 \leq \kappa \leq 1.597$ and unstable for $\kappa \geq 1.598$. For $N \geq 9$ the configuration is unstable for all κ; except $\kappa = 0$

*Thomson calculated an extraneous eigenfrequency larger than Ω_0 for $\kappa = 0$ and N = 5 and 6 due to misusing the second of the two linear integral invariants (3.17) and (3.19).

is a neutral stability point for N = 9. Comparing with Fig. 2 we see that N = 8, 9, 10 for $\gamma_0 = 1$ are comparable to N = 6, 7, 8 respectively, for $\gamma_0 = 0$. That is, the presence of a center vortex ($\gamma_0 = 1$) allows the presence of <u>two additional circle vortices</u> without appreciably changing the stability characteristics. However, for N = 2, this (N + 1)-configuration is <u>unstable</u> for all κ; for N = 3, this configuration is <u>unstable</u> for $0 \leq \kappa \leq 3.45$ and stable for $\kappa \geq 3.50$. Thus for an <u>intermediate number</u> of circle vortices this (N + 1)-configuration is <u>stable</u> for all κ, i.e., for N = 4, 5, 6 and 7.

Comparison with $\gamma_0 = 0$ implies that the instability for <u>large</u> N (> 6) means an unstable circle vortex. The instability for <u>small</u> N (< 4) means an <u>unstable center vortex</u>; this is clearly seen by geometrical considerations for N = 2 and $\gamma_0 \longrightarrow 0$ (cf. Fig. 6, $\kappa = 0$). These characteristics become more apparent in the following sub-sections.

In Fig. 5 the variation of the normal frequencies with respect to κ for N = 8 is shown. The equilibrium frequency $\Omega = \Omega_0 + \gamma_0\omega_0(\gamma_0 = 1)$ is marked.

3.4 (N + 1) Vortices, General Case ($-\infty < \gamma_0 < \infty$).

We are now in a position to describe the general case where $N \geq 2$, $0 \leq \kappa < \infty$ and $-\infty < \gamma_0 < \infty$. The simplest approach is, first, to study the special case of logarithmic vortices ($\kappa = 0$) which is representative of the γ_0 variation

of the eigenvalues for all κ.

In Fig. 6 the variation of the maximum real eigenvalues with respect to γ_o for $\kappa = 0$ and $2 \leq N \leq 10$ is shown. Two distinct families of real eigenvalues are evident: one family, which yields (for different N) stable motions for γ_o greater than the neutral stability point, is pertinent for the N circle vortices; and the other family, which yields (for different N) stable motions for γ_o less than the neutral stability point, is pertinent for the center vortex. Note that for N = 2 the unstable range includes $\gamma_o = 0$, i.e. unstable center vortex. For example, for N = 4 the region of stable motions is $-0.5 < \gamma_o < 2.25$; for $\gamma_o \leq -0.5$ a circle vortex is unstable and for $\gamma_o \geq 2.25$ the center vortex is unstable. In Fig. 7 the variation of the normal frequencies with respect to γ_o for $\kappa = 0$ and $2 \leq N \leq 6$ is shown. The equilibrium frequency Ω varies linearly with γ_o as well as with N.

The linearized stability of (N + 1) vortices over the entire range of the three parameters N, κ and γ_o are summarized in two figures, Fig. 8 and Fig. 9: 1) Fig. 8 shows the neutral stability curves for the circle vortices where the stable region is to the right of each curve and 2) Fig. 9 shows the neutral stability curves for the center vortex where the stable region is to the left of each curve. Both figures must be superposed in order to determine the regions of stability (or instability) for each N. For N = 2 and 3 there

is a semi-infinite range of stability for large negative γ_o, i.e. the circle vortices are stable for all γ_o. For $N \geq 4$ the vortex configuration is stable in a finite range of γ_o and the size of this range increases rapidly for large N; this is shown in Table I. for $\kappa = 0$.

3.5 (N + 1) Vortices with a Fixed Center Vortex.

The vortex configurations studied in the previous sections were free vortices without external constraints. In this section we consider the effect of holding the center vortex fixed at the center instead of being free to move. Then Eqs. (3.5) - (3.8) are modified by the additional conditions.

$$\tilde{x}_{\odot}^{(1)} = \tilde{y}_{\odot}^{(1)} = 0 \quad \text{and} \quad \dot{\tilde{x}}_{\odot}^{(1)} = \dot{\tilde{y}}_{\odot}^{(1)} = 0 .$$

The number of degrees of freedom is 2N instead of 2(N + 1) for the free case. Eqs. (3.7) and (3.8) are identically zero; and Eqs. (3.5) and (3.6) now have the circulant property discussed in §3.1 for the case ($\gamma_o = 0$) without a

Table I

Range of Exponential Stability for Free Center Vortex Case, $\kappa = 0$

N	Lower Stability Limit*	Upper Stability Limit**
2	$\gamma_o = -\infty$	$\gamma_o = -1.25$
3	$-\infty$	1
4	-.5	2.25
5	-.5	4
6	-.25	6.25
7	0	9
8	.5	12.25
9	1	16
10	1.75	20.25
11	2.5	25
12	3.5	30.25
13	4.5	36
14	5.75	42.25
15	7	49

* A circle vortex is unstable for γ_o less than the lower stability limit.

** The center vortex is unstable for γ_o greater than the upper stability limit.

center vortex. Again the eigenvalue calculation can be reduced to solving a quadratic equation, the only difference from the case ($\gamma_o = 0$) being a modification of right diagonal terms in the submatrix C, i.e. the coefficient of $r_k^{(1)}$ is $(\Omega + C_o + \gamma_o D_o)$ instead of $(\Omega_o + C_o)$ for $\gamma_o = 0$ (cf. [2]).

The main results are: 1) for $N \geq 4$ the instability characteristics of the circle vortices are the same for fixed and free center vortex, i.e. same real eigenvalues, 2) for N = 2 and 3 the <u>circle vortices have a semi-infinite unstable range for negative</u> γ_o; for example for $\kappa = 0$ the circle vortices are stable for N = 2 over the range $\gamma_o > -0.25$ and for N = 3 over the range $\gamma_o > -0.5$ (cf. Table I for free center vortex) and 3) the equilibrium frequency Ω is not an eigenfrequency if the center vortex is fixed; and 2(N - 2) of the eigenvalues are the same as the eigenvalues for the free (center vortex) case, including the double zero roots. This is an example in which constraints have introduced some instability.

3.6 Multiple Eigenfrequencies

Over the entire range of the parameters N, κ and γ_o the zero eigenvalue is at least a double root; on the neutral stability curves shown in Figs. 8 and 9 there are quadruple zero roots. In addition there are a countable infinity of multiple non-zero eigenfrequencies, especially for $\kappa = 0$. By standard methods [12] we can demonstrate that the multiple non-zero eigenvalues do not produce solutions of the form $t^{\mu-1} \cdot e^{\lambda t}$, where μ is the multiplicity. However the multiple zero roots require some attention.

There are two linear invariants associated with the system of equations (3.5) to (3.8) inclusive. Either by summing the N equations (3.5) or by linearizing the non-linear integral invariant (A5.2), we get

$$\sum_{k=1}^{N} \dot{r}_k^{(1)} = 0 \tag{3.16}$$

and

$$\sum_{k=1}^{N} r_k^{(1)} = \text{const.} = 0 \tag{3.17}$$

The constant in (3.17) can be set equal to zero by an appropriate choice of the initial equilibrium radius. Either by summing the N equations (3.6) or by linearizing the non-linear integral invariant (A6.1), we get

$$(3.18) \quad \sum_{k=1}^{N} \dot{\theta}_k^{(1)} = -\left(\sum_{k=1}^{N} r_k^{(1)}\right)\left[\Omega + C_o + \gamma_o D_o + \sum_{\substack{i \neq k \\ i=1}}^{N} C_{ki}\right]$$

$$= \text{const.} = 0$$

The constant in (3.18) is zero only if the constant in (3.17) is zero. Then (3.18) implies

$$(3.19) \qquad \sum_{k=1}^{N} \theta_k^{(1)} = \text{const.}$$

Since the second linear invariant (3.19) is a consequence of the first linear invariant (3.17), the constant in (3.19) cannot be set equal to zero for all possible initial conditions.

For $\gamma_o = 0$ the linear invariants (3.17) and (3.19) can be associated with the double zero eigenvalues, eliminating unstable solutions of the form $t^{\mu-1}$ where μ is the multiplicity. But at a neutral stability point such as $\kappa = 0$ for $N = 7$, the quadruple zero eigenvalues yields a linearly time-increasing solution.

The linear invariants (3.17) and (3.19) are apparently valid for $\gamma_o \neq 0$; but since these invariants are independent of the position of the center vortex, we suspect that a linearized description of the motion may not be strictly valid. If these invariants cannot be invoked, the double zero roots yield an algebraic instability proportional to t over the entire range of parameters (except for $\gamma_o = 0$).

The implication is that the center vortex motion is inherently non-linear. Thus we must study the non-linear initial value problem in order to determine the range of validity of the linearized stability analysis. In deriving the linearized equations (3.5) to (3.8), some peculiarities of the center vortex were already evident: first, mixed cartesian and polar coordinates were needed to describe the motion of the center and circle vortices; and second, the center vortex terms introduced periodic coefficients which fortuitously could be made constant by transformation to an appropriate rotating coordinate system.

Appendix I. Integrals of the Motion.

The number of integral invariants or constants of the motion is the same as for the equations of motion of logarithmic vortices (cf. Lamb [13]). Likewise the kinematic equations of motion for geostrophic vortices can be written in a more symmetric form than those given in §2

$$\gamma_k(\dot{x}_k, \dot{y}_k) = \left(\frac{\partial W}{\partial y_k}, -\frac{\partial W}{\partial x_k}\right) \tag{A1}$$

where the Kirchhoff function W for geostrophic vortices is

$$W = \frac{1}{4\pi} \sum_{\substack{i \neq j \\ i,j=1}}^{N} \gamma_i \gamma_j \, K_o(\rho_{ij}) \tag{A2}$$

and $\rho_{ij} = \kappa[(x_i - x_j)^2 + (y_i - y_j)^2]^{1/2}$; for $\kappa = 0$, W reduces to the original logarithmic expression of Kirchhoff. In polar coordinates (A1) becomes

$$\gamma_k \dot{r}_k = \frac{1}{r_k} \frac{\partial W}{\partial \theta_k} \tag{A3.1}$$

$$\gamma_k r_k \dot{\theta}_k = -\frac{\partial W}{\partial r_k} \ . \tag{A3.2}$$

The integral invariants are:

1) $\dot{W} = 0$ and W is an integral invariant. This follows from the symmetry of W in i and j; and because W is not an explicit function of time and is only a function of the distance between vortices ρ_{ij}.

2) From (A1) and the translational invariance of W

(A4.1)
$$\sum_{k=1}^{N} \gamma_k \dot{x}_k = \sum_{k=1}^{N} \frac{\partial W}{\partial y_k} = 0$$

(A4.2)
$$\sum_{k=1}^{N} \gamma_k \dot{y}_k = - \sum_{k=1}^{N} \frac{\partial W}{\partial x_k} = 0$$

and, for example, the centroid is an integral invariant

(A4.3)
$$\frac{\sum_{k=1}^{N} \gamma_k x_k}{\sum_{k=1}^{N} \gamma_k} = \text{const.}, \qquad \frac{\sum_{k=1}^{N} \gamma_k y_k}{\sum_{k=1}^{N} \gamma_k} = \text{const.}$$

if $\sum_{k=1}^{N} \gamma_k \neq 0$; for $\sum_{k=1}^{N} \gamma_k = 0$, the numerators are still constants but the physical interpretation is less apparent.

3) From (A3.1) and the rotational invariance of W

(A5.1)
$$\sum_{k=1}^{N} \gamma_k r_k \dot{r}_k = \sum_{k=1}^{N} \frac{\partial W}{\partial \theta_k} = 0$$

and

(A5.2)
$$\sum_{k=1}^{N} \gamma_k r_k^2 = \text{const.}$$

The invariant (A5.2) for this kinematic system is comparable to the momentum invariant in dynamical systems.

4) From (A3.2)

$$\text{(A6.1)} \qquad \sum_{k=1}^{N} \gamma_k r_k^2 \dot{\theta}_k = - \sum_{k=1}^{N} r_k \frac{\partial W}{\partial r_k}$$

$$= \frac{1}{4\pi} \sum_{\substack{i \neq k \\ i,k=1}}^{N} \gamma_i \gamma_k \rho_{ik} K_1(\rho_{ik})$$

$$= \text{const.}$$

since the right hand side of (A6.1) has the same symmetry properties as W. For $\kappa = 0$, (A6.1) reduces to the classical result,

$$\text{(A6.2)} \qquad \sum_{k=1}^{N} \gamma_k r_k^2 \dot{\theta}_k = \frac{1}{4\pi} \sum_{\substack{i \neq k \\ i,k=1}}^{N} \gamma_i \gamma_k$$

Acknowledgements

We wish to thank A. Kasahara and his colleagues at the National Center for Atmospheric Research, Boulder, Colorado for their encouragement and interest in this work.

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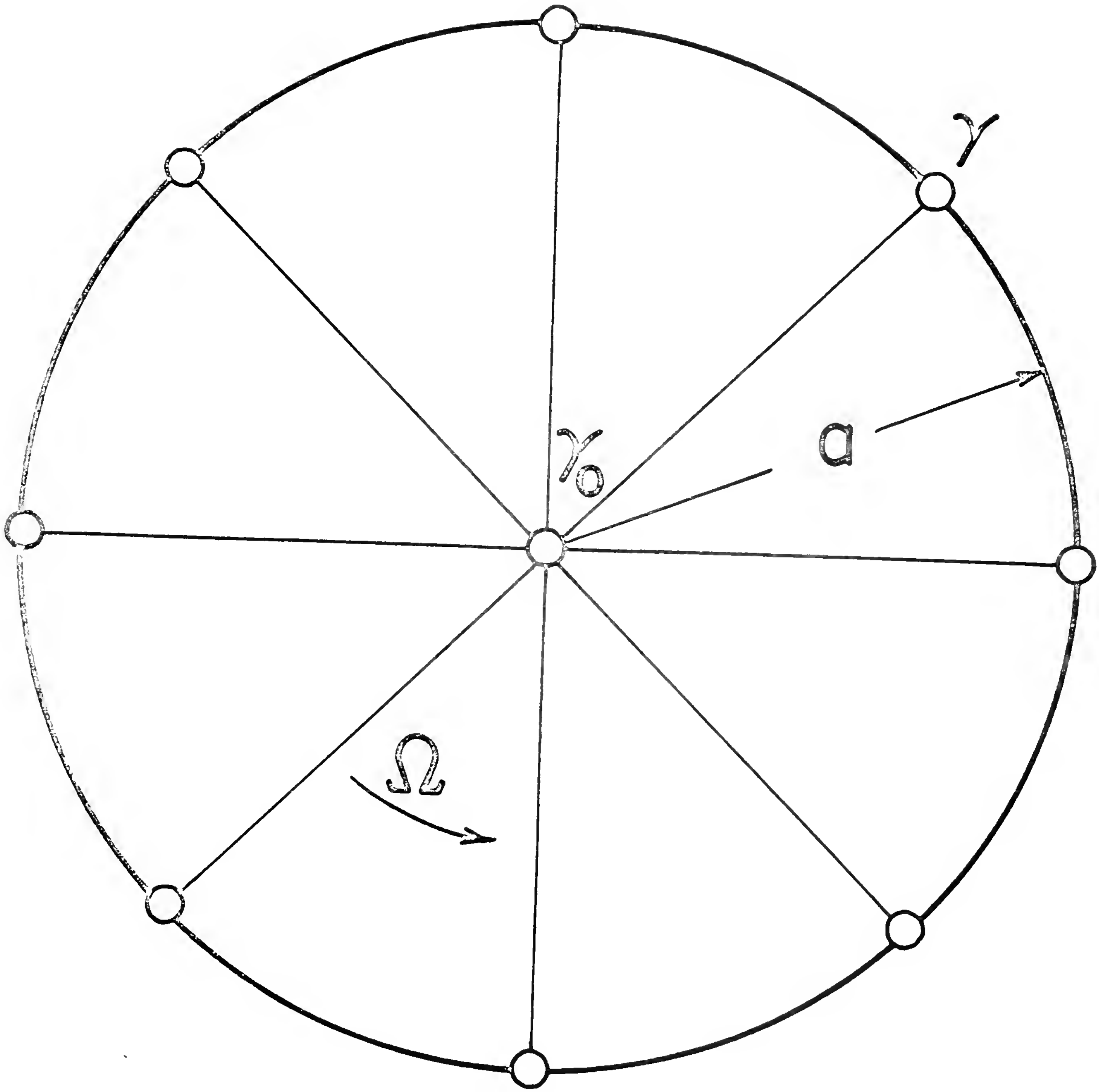

Fig. 1 Equilibrium Configuration for N = 8

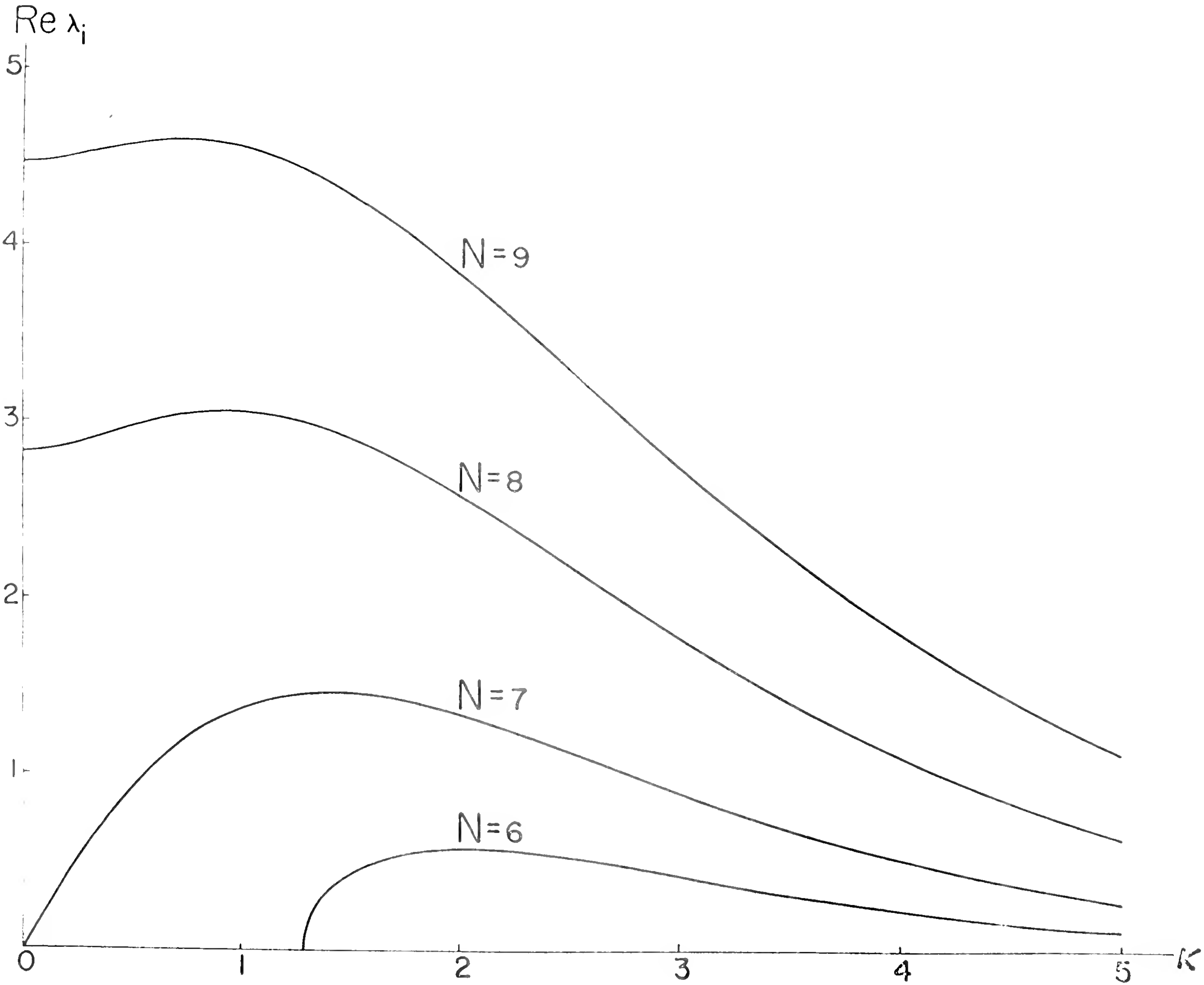

Fig. 2 Unstable Roots for γ_o = 0 (No center vortex)

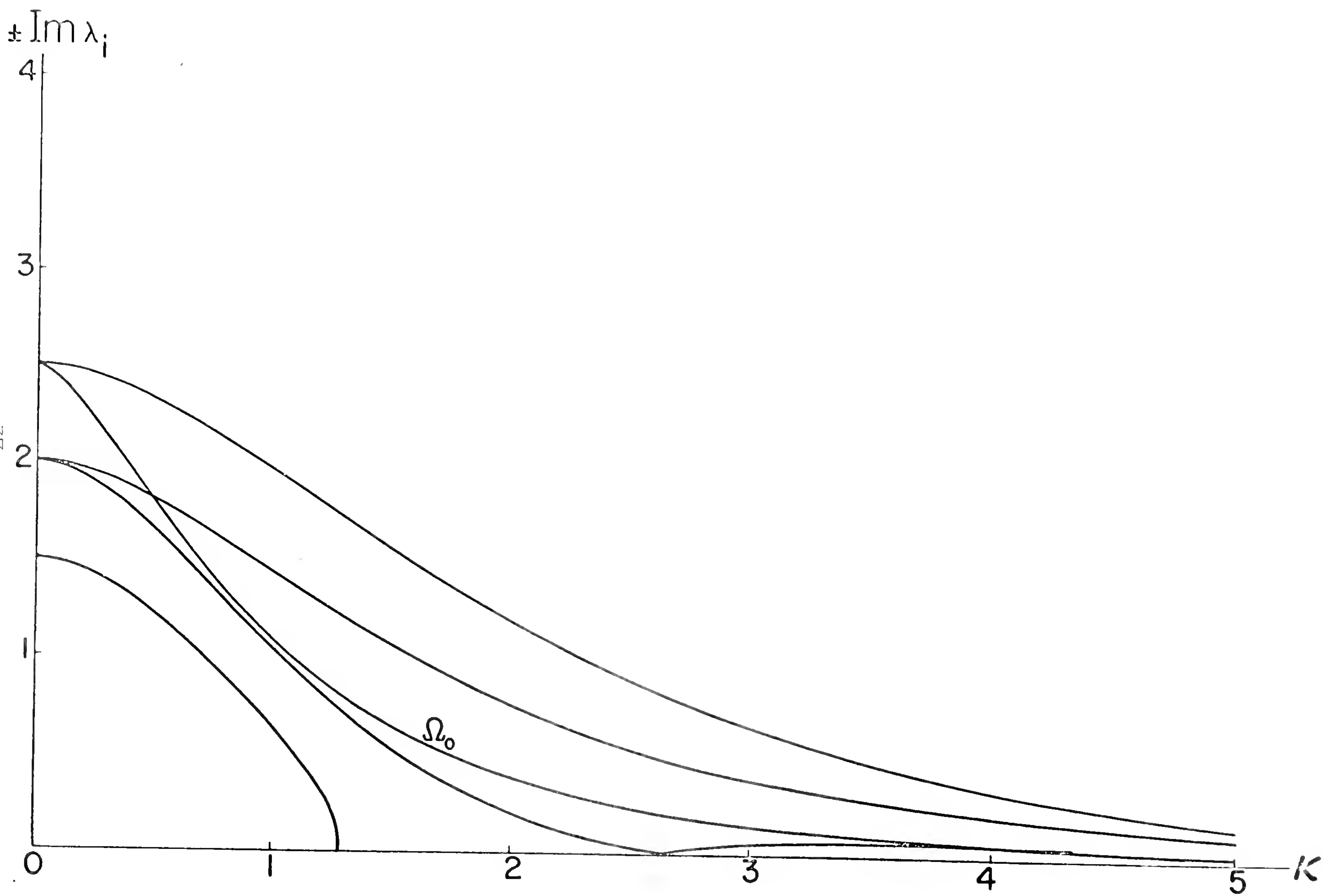

Fig. 3 Eigenfrequencies for $\gamma_o = 0$, N = 6

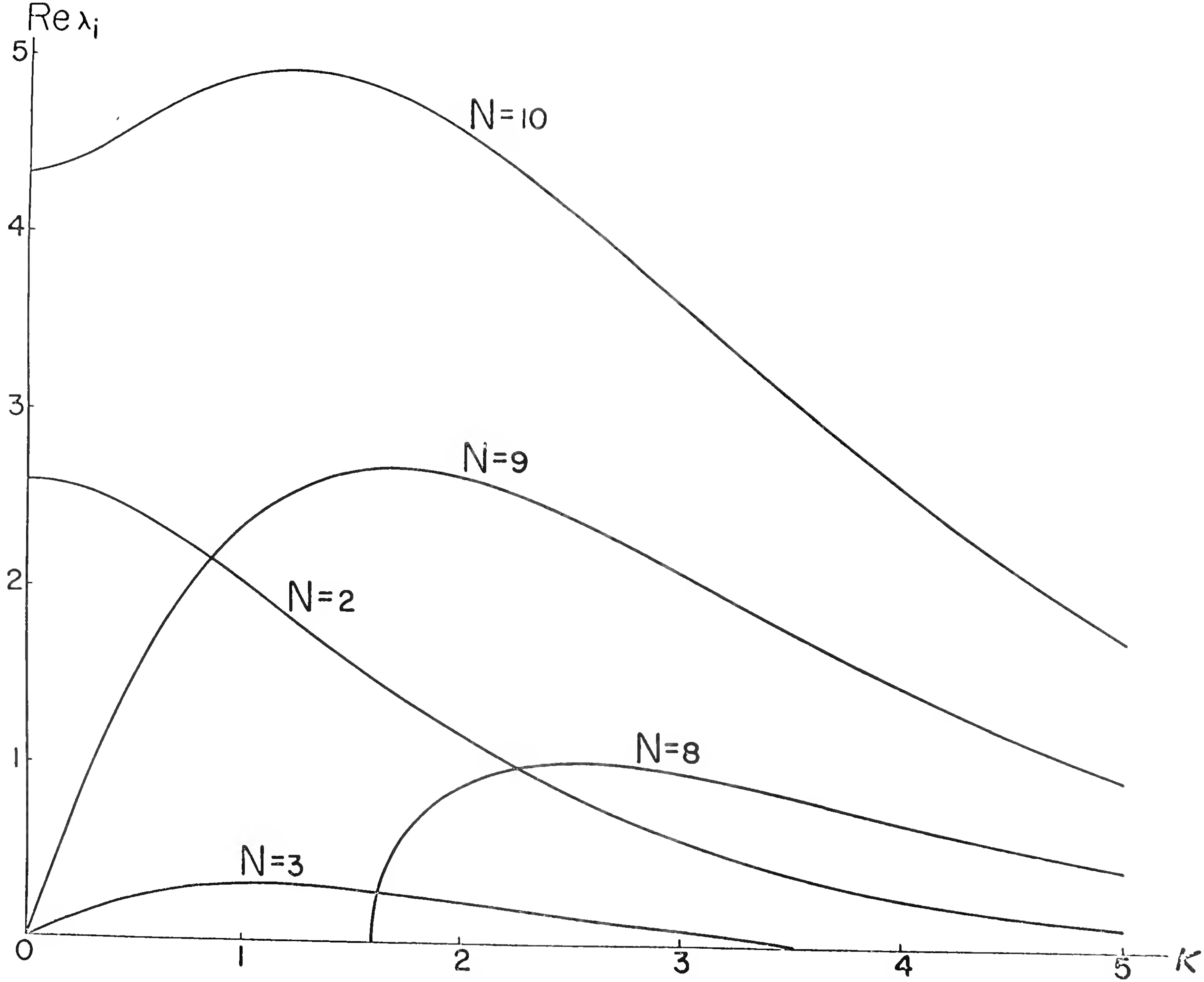

Fig. 4 Unstable Roots for γ_o = 1

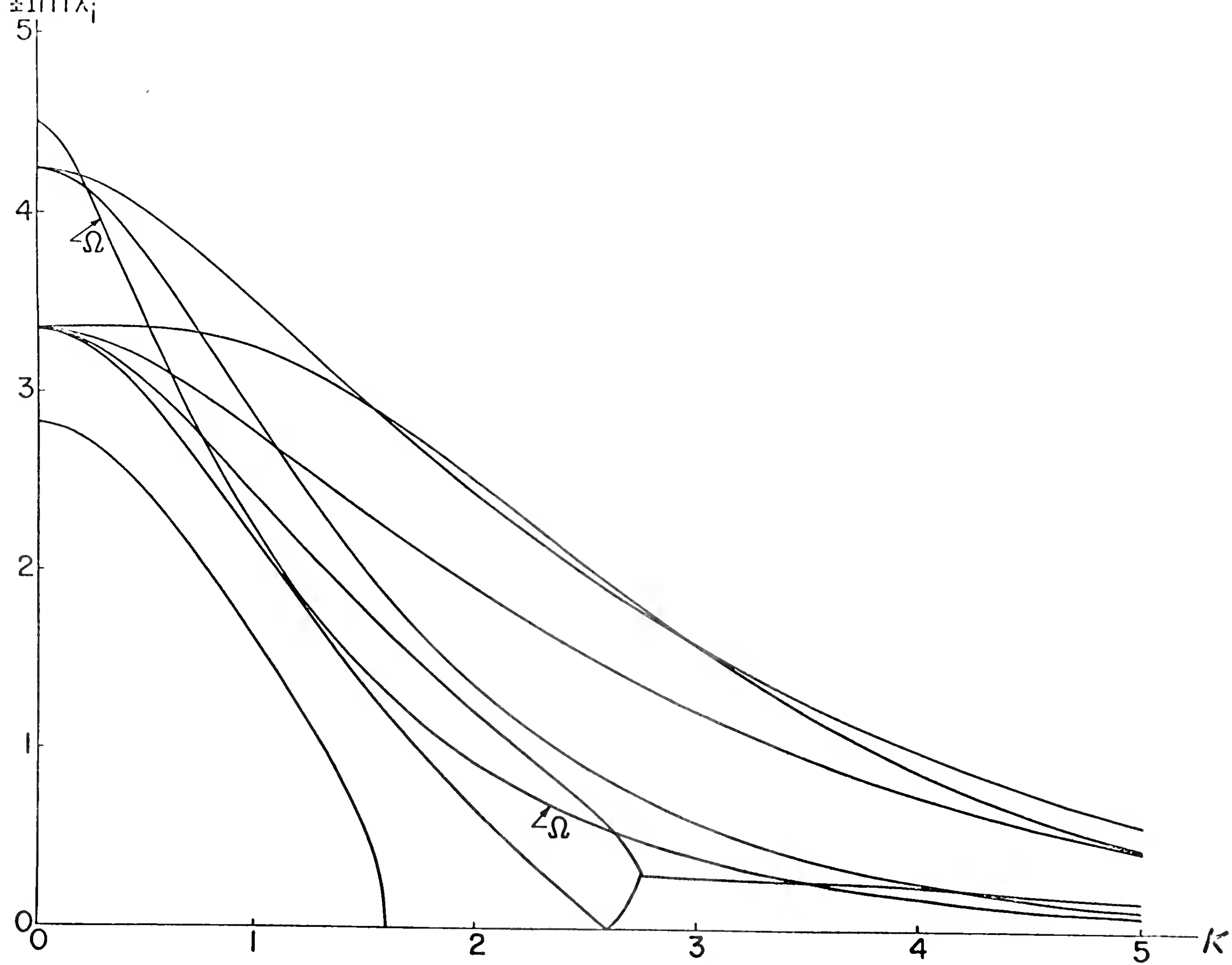

Fig. 5 Eigenfrequencies for $\gamma_o = 1$, N = 8

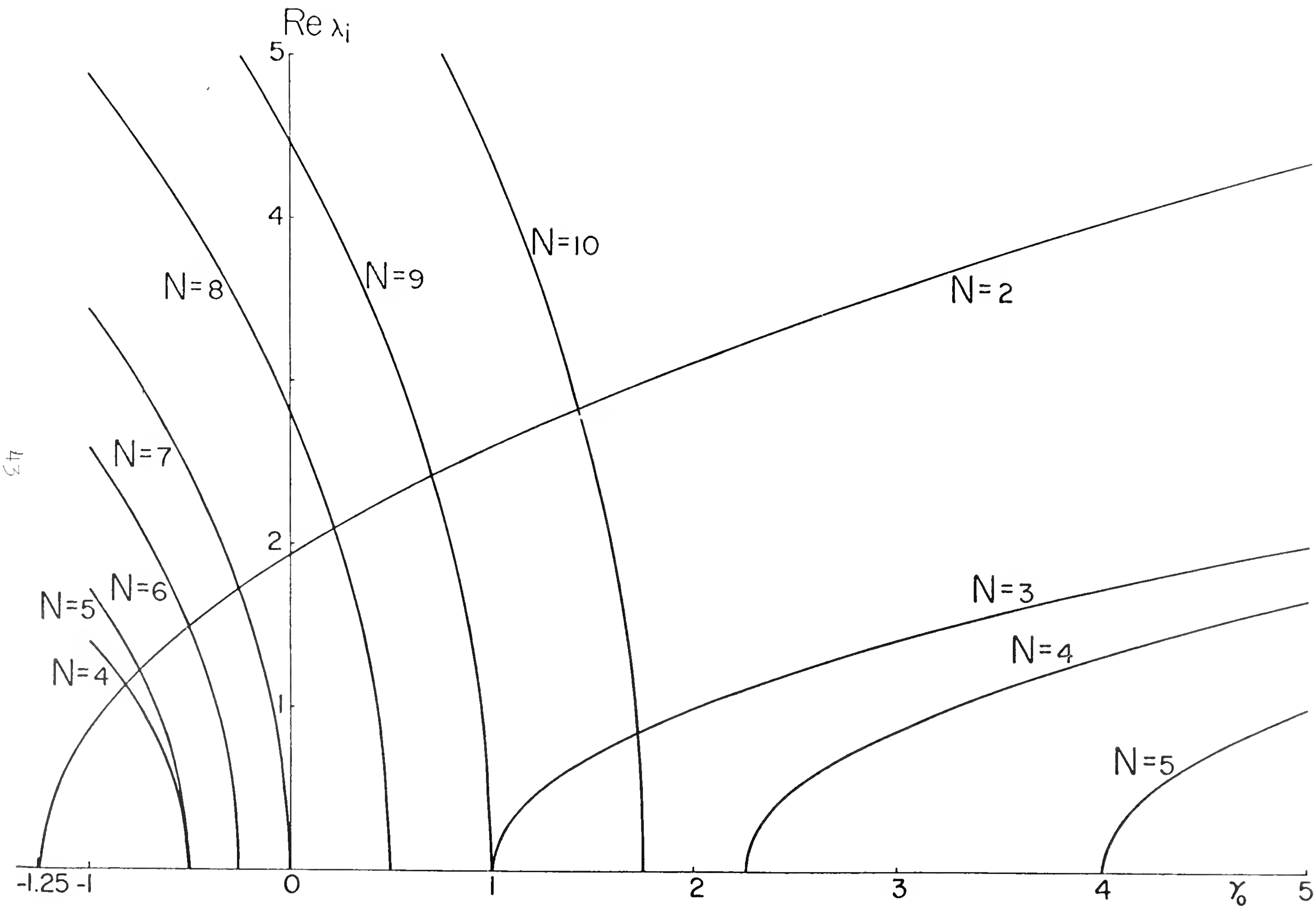

Fig. 6 Unstable Roots for $\kappa = 0$

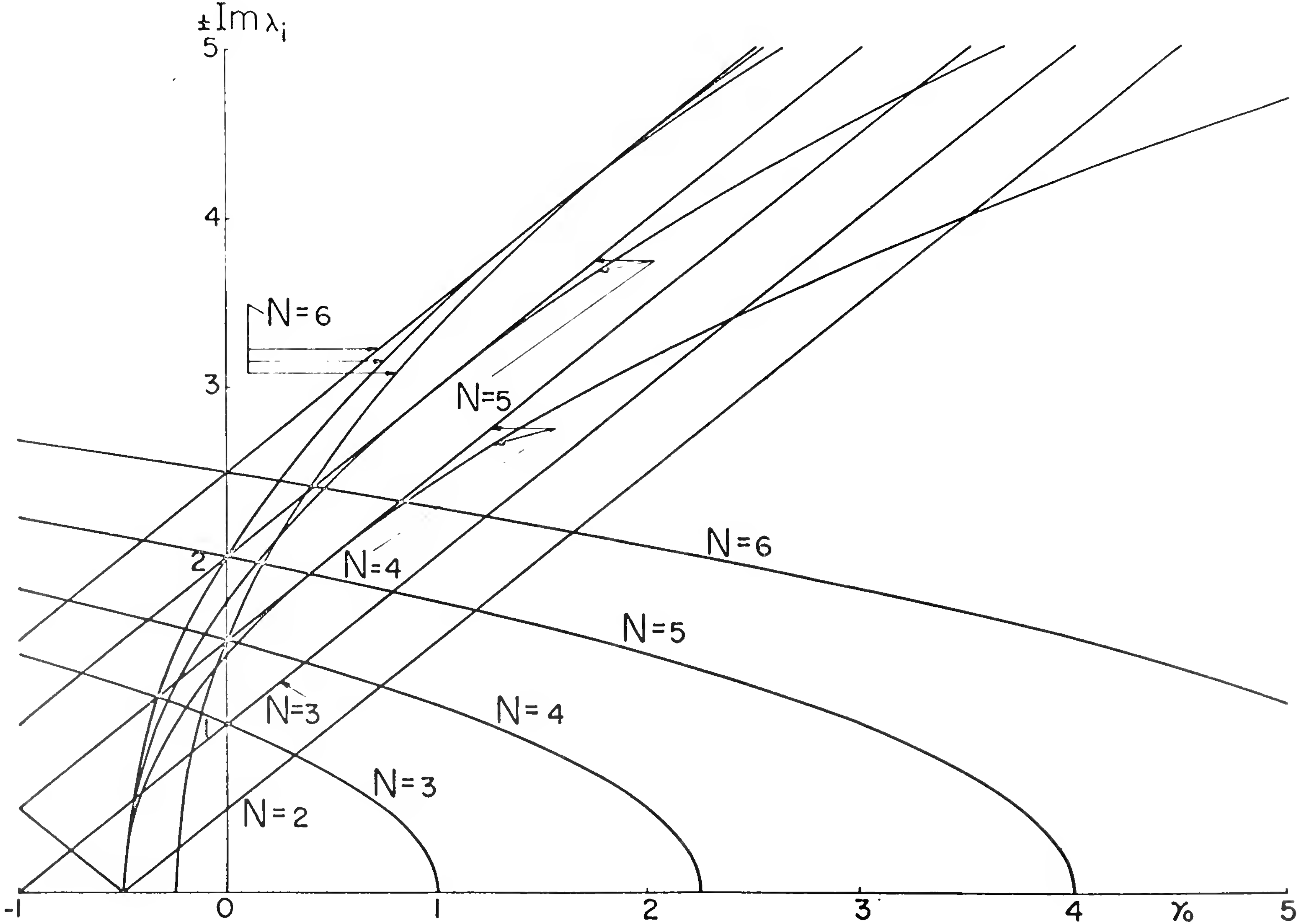

Fig. 7 Eigenfrequencies for $\kappa = 0$

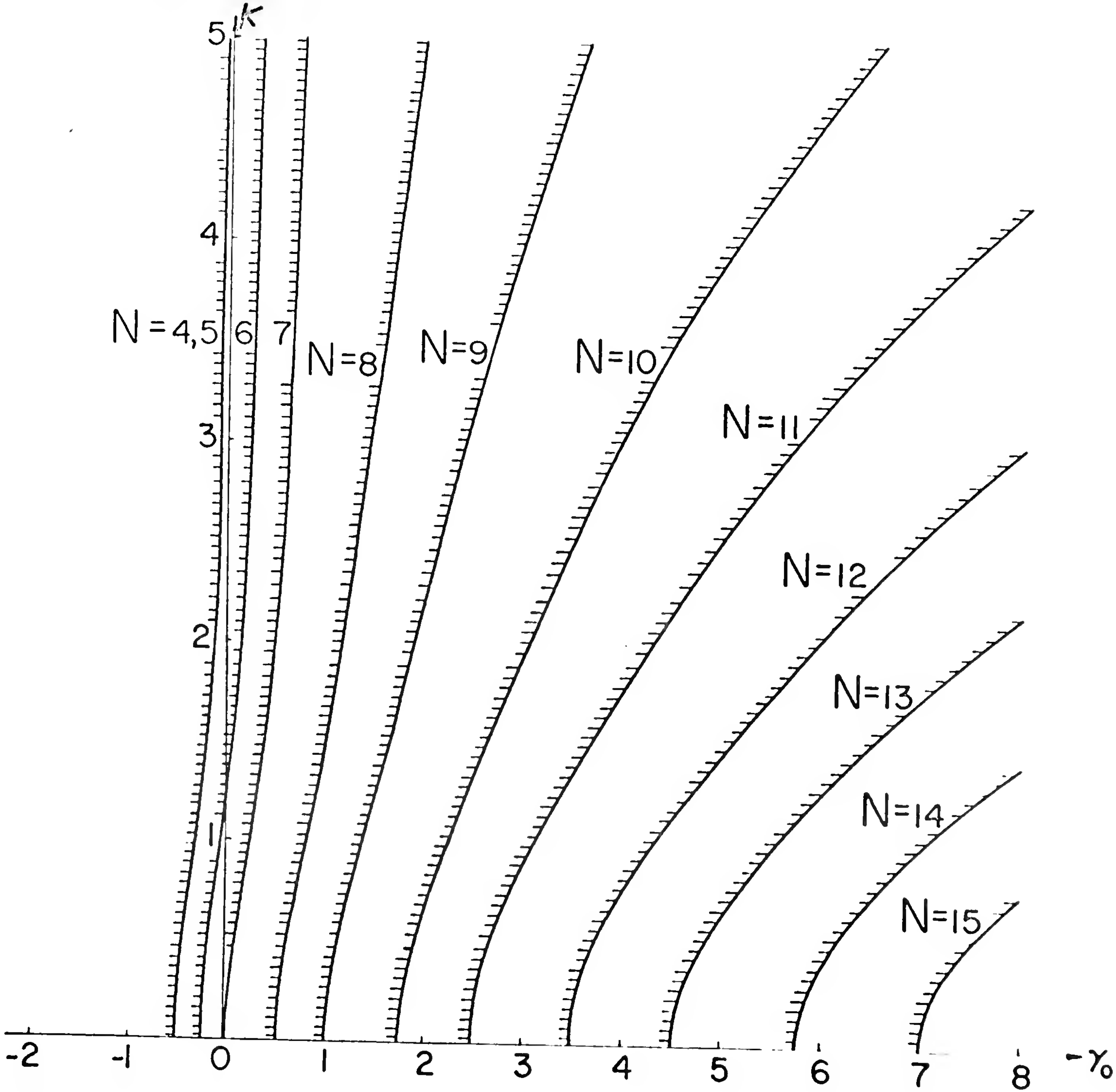

Fig. 8 Neutral Stability Curves for Circle Vortices (Stable region on right side of each curve)

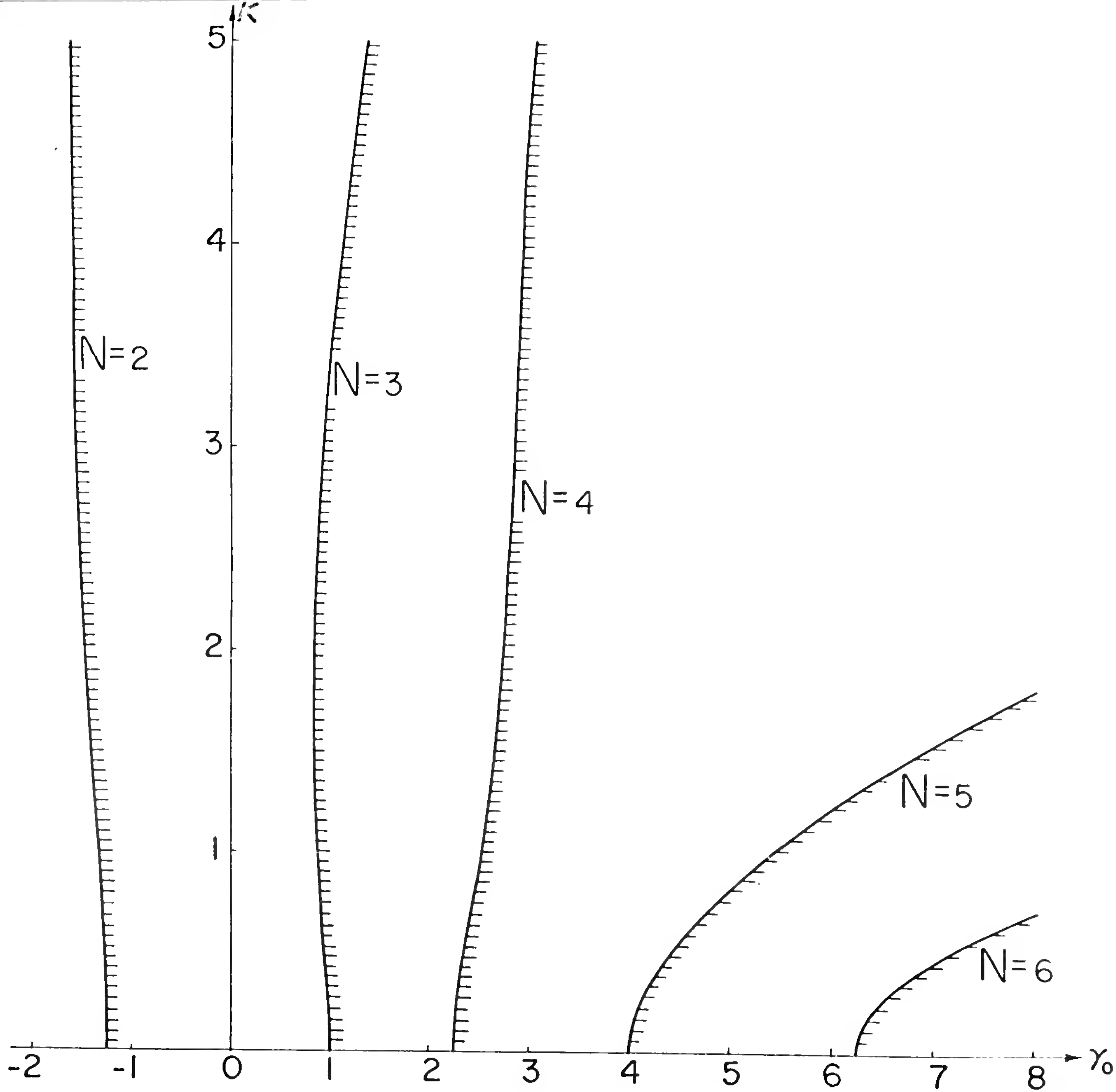

Fig. 9 Neutral Stability Curves for Center Vortex (Stable region on left side of each curve)

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Interacting Motion of Rectilinear Geostrophic Vortices
I. Linear Stability Analysis

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report January 1968

5. AUTHOR(S) (Last name, first name, initial)

Morikawa, George K. and Swenson, Eva V.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
January 1968	56	13

8a. CONTRACT OR GRANT NO. Nonr-285(55)	9a. ORIGINATOR'S REPORT NUMBER(S) IMM 364
b. PROJECT NO. NR 062-160	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) none
d.	

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13 ABSTRACT

The kinematic motion of (N+1) geostrophic (Bessel) vortices is studied. Initially the vortices are positioned near a uniformly rotating equilibrium configuration which consists of N vortices of equal strength γ, equally spaced on a circle of radius a, and one vortex of strength γ_o at the center of the circle. One length scale κ^{-1} is associated with each of the vortices, the limiting case $\kappa = 0$ being the classical logarithmic vortex. The exponential stability of this equilibrium configuration is studied over the range of the three parameters: $N \geq 2$, $0 \leq (\kappa a) < \infty$ and $-\infty < (\gamma_o/\gamma) < \infty$. The need for a supplementary nonlinear study is discussed.

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